Announcements

1) week after spring break
   = (almost) all review.

2) In-class exam week
   after week after spring
   tentatively Tuesday

Properties of Series (sums, scalars)

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

both converge and $c \in \mathbb{R}$.

Then

1) $$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,$$

hence convergent

2) $$\sum_{n=1}^{\infty} (c a_n) = c \sum_{n=1}^{\infty} a_n$$

hence convergent,

proof: both properties follow immediately from the corresponding properties for sequences by considering the partial sums $\square$

Note: (products & rearrangements)

You don't get any formula for the product of two convergent for free!

Need additional assumptions.

Theorem: (comparison) Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence with $\boxed{a_n \geq 0}$ for all $n \in \mathbb{N}$.

If $b_n \geq a_n \ \forall\, n \in \mathbb{N}$, then

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

Proof:

1) Let $S_k = \sum_{n=1}^{k} a_n$. Since

$a_n \geq 0$, $S_{k+1} \geq S_k$.

Also, $$\sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} b_n \quad (\text{since } b_n \geq a_n \ \forall n \in \mathbb{N})$$

$$\leq \sum_{n=1}^{\infty} b_n \quad (\text{since } b_n \geq 0),$$

$$= L, \text{ a real number.}$$

Hence, $(S_k)_{k=1}^{\infty}$ is a bounded, monotonic sequence, hence converges

2) Same proof as 1),

$$S_k = \sum_{n=1}^{k} b_n \geq \sum_{n=1}^{k} a_n = T_k$$

Since $a_n \geq 0 \ \forall n \in \mathbb{N}$, if $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{k \to \infty} T_k = \infty$

$\Rightarrow \lim_{k \to \infty} S_k = \infty \Rightarrow$

$\sum_{n=1}^{\infty} b_n$ diverges. $\square$

Ridiculous Example

Show $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)+2}{n^3-\frac{1}{2}}$

converges. Use comparison!

$\sin^2(n) \leq 1$, so

$\sin^2(n)+2 \leq 1+2=3$

$$\boxed{n^3-\frac{1}{2} \overset{?}{\geq} n^3-\frac{n^3}{2}}$$

since $n \geq 1$, $n^3 \geq 1$

So then

$$n^3 - \frac{1}{2} \geq n^3 - \frac{n^3}{2} = \frac{n^3}{2},$$

which gives us

$$a_n = \frac{\sin^2(n) + 2}{n^3 - \frac{1}{2}} \leq \frac{3}{\left(\frac{n^3}{2}\right)} = \frac{6}{n^3} = b_n$$

$$\sum_{n=1}^{\infty} \frac{6}{n^3} = 6 \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{converges}$$

since $3 > 1$ (p-test)

This shows

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)+1}{n^3 - 1/2} \quad \text{converges}$$

by comparison.

Definition: (absolute/cond. conv.)

(leads toward when we can say something about the product of two series)

A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

A series

$\sum_{n=1}^{\infty} a_n$ is said to converge conditionally if $\sum_{n=1}^{\infty} a_n$ converges

but $\sum_{n=1}^{\infty} |a_n|$ does not converge.

Proposition: (abs $\Rightarrow$ conv)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then

$\sum_{n=1}^{\infty} a_n$ converges.

proof: write $\sum_{n=1}^{\infty} a_n$ as the sum of two convergent series

$$a_n = |a_n| - |a_n| + a_n$$

$$= \underbrace{(|a_n| + a_n)}_{\text{converges as a series?}} - \underbrace{|a_n|}_{\text{converges as a series}}$$

Use comparison:

$$0 \leq |a_n| + a_n \leq 2|a_n|$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, so does

$$\sum_{n=1}^{\infty} 2|a_n|$$

So by comparison,

$$\sum_{n=1}^{\infty} \left(|a_n| + a_n \right) \text{ converges}$$

We know $\displaystyle\sum_{n=1}^{\infty} -|a_n| = -\sum_{n=1}^{\infty} |a_n|$

converges by assumption, hence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\left(|a_n| + a_n \right) - |a_n| \right)$$

$$= \sum_{n=1}^{\infty} \left(|a_n| + a_n \right) - \sum_{n=1}^{\infty} |a_n| ,$$

which gives the result that

$$\sum_{n=1}^{\infty} a_n \quad \text{converges.} \qquad \square$$

What is an example of a conditionally convergent series?

Theorem. (alternating series)

Suppose

1) $b_n \geq 0 \quad \forall\, n \in \mathbb{N}$

and

2) $b_n \geq b_{n+1}$ for all $n \in \mathbb{N}$.

Then $\boxed{\sum_{n=1}^{\infty} (-1)^n b_n \text{ converges.}}$

proof: Idea is to set

$$S_k = \sum_{n=1}^{k} (-1)^n b_n$$ and

show $(S_k)_{k=1}^{\infty}$ is Cauchy.

Consider

$|S_k - S_m|$ and suppose, without loss of generality, that

$k > m$,

$$|S_k - S_m|$$

$$= \left| \sum_{n=1}^{k} (-1)^n b_n - \sum_{n=1}^{m} (-1)^n b_n \right|$$

$$= \left| \sum_{n=m+1}^{k} (-1)^n b_n \right|$$

$$= \left| (-1)^{m+1} b_{m+1} + (-1)^{m+2} b_{m+2} + - + (-1)^k b_k \right|$$

$$= \left| b_{m+1} - b_{m+2} + b_{m+3} + - + (-1)^{k-m-1} b_k \right|$$

by multiplying all terms

by $(-1)^{-1-m}$

Claim; Let m be arbitrary

$$0 \leq b_{m+1} - b_{m+2} + \cdots + (-1)^{k-m-1} b_k$$

$$\leq b_{m+1}.$$

We prove this via induction on $k-m$. Suppose $k=m+1$.

Then since $b_n \geq b_{n+1} \geq 0 \;\forall n \in \mathbb{N}$,

$$0 \leq b_{m+1} - b_{m+2} \leq b_{m+1}.$$

Then also

$b_{m+1} - b_{m+2} + b_{m+3}$

$= b_{m+1} - \left(b_{m+2} - b_{m+3} \right)$

and since $0 \leq b_{m+2} - b_{m+3} \leq b_{m+2}$

$b_{m+1} \geq b_{m+1} - b_{m+2} + b_{m+3} \geq b_{m+1} - b_{m+2} \geq 0$

So assume the statement is true for $k - m = t \geq 1$.

Suppose $k - m = t+1$

Then

$$b_{m+1} - b_{m+2} + \cdots + (-1)^{k-m-1} b_k$$

$$= b_{m+1} - \left(b_{m+2} - b_{m+3} + \cdots + (-1)^{k-m} b_k\right)$$

and by induction,

$$0 \le \underbrace{b_{m+2} - b_{m+3} + \cdots + (-1)^{k-m} b_k}_{t \text{ terms}} \le b_{m+2}, \text{ so}$$

$$b_{m+1} \ge b_{m+1} - \left(b_{m+2} - b_{m+3} + \cdots + (-1)^{k-m} b_k\right)$$

$$\ge b_{m+1} - b_{m+2} \ge 0$$

and so the induction is complete

We then know

$$|S_k - S_m| \leq b_{m+1},$$

and since $b_{m+1} \longrightarrow 0$ as $m \to \infty$, $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ with $b_{m+1} < \varepsilon\ \forall\ m+1 \geq N$.

Hence, $\forall\ k, m \geq N-1$,

$|S_k - S_m| < \varepsilon$, and so

$(S_k)_{k \in \mathbb{N}}$ is Cauchy. Therefore, $(S_k)_{k \in \mathbb{N}}$ converges. $\square$